Journal of Approximation Theory **101**, 221–239 (1999)
Article ID jath.1999.3368, available online at http://www.idealibrary.com on IDEAL®

On a Conjecture Concerning Strong Unicity Constants

Wolfgang Gehlen

Mathematik, Fachbereich 4, Universität Trier, D-54286 Trier, Germany
E-mail: gehlen@math39.uni-trier.de

Communicated by András Kroó

Received March 9, 1998; accepted in revised form March 25, 1999

Let $f \in C[-1, 1]$ be real-valued. We consider the sequence of strong unicity constants $(\gamma_n(f))_n$ induced by the polynomials of best uniform approximation of f. It is proved that $\liminf_{n \to \infty} \gamma_n(f) = 0$, whenever f is not a polynomial.

1. STATEMENT OF THE RESULT AND NOTATIONS

For a given real-valued function $f \in C[-1, 1]$ we denote by q_n^*, $n \in \mathbb{N}_0$, its best uniform approximation in the set P_n of algebraic polynomials of degree at most $n \in \mathbb{N}_0$:

$$\|f - q_n^*\| = \min_{q \in P_n} \|f - q\| = \min_{q \in P_n} \{ \max_{x \in [-1, 1]} |f(x) - q(x)| \}.$$

In this situation the following strong uniqueness theorem holds.

Theorem A (Newman and Shapiro [11, Theorem 4]). *For each $n \in \mathbb{N}_0$ there exists a constant $C = C_n(f) > 0$ such that*

$$\|f - q\| \geqslant \|f - q_n^*\| + C \|q - q_n^*\| \qquad \textit{for all} \quad q \in P_n. \tag{1}$$

Definition. For each $n \in \mathbb{N}_0$ the largest constant C such that (1) holds is called the *strong unicity constant* and will be denoted by $\gamma_n(f)$. We put $M_n(f) := 1/\gamma_n(f)$.

Poreda [12] raised the question to describe the behaviour of the sequence $(M_n(f))_n$ for a given function f, and there are various results on this problem [2–9, 13].

If, for instance, $f \in P_m$ is a polynomial it is easy to see that $M_n(f) = 1$ for all $n \geqslant m$. In this paper we shall prove the following conjecture of Henry and Roulier [6].

0021-9045/99 $30.00

THEOREM. *If f is not a polynomial, then we have*

$$\limsup_{n\to\infty} M_n(f)=\infty.$$

We note that the proof presented here will not provide any concrete estimate for the sequence $M_n(f)$.

To prove this result let

$$E_n=E_n(f):=\{x\in[-1,1]: |f(x)-q_n^*(x)|=\|f-q_n^*\|\}, \qquad n\in\mathbb{N}_0$$

and

$$\sigma_n(x)=\sigma_n(f,x):=\operatorname{sign}(f-q_n^*)(x), \qquad n\in\mathbb{N}_0.$$

The strong unicity constant $\gamma_n(f)$ can be characterized in terms of E_n and $\sigma_n(x)$.

THEOREM B (Bartelt and McLaughlin [1] or [3, p. 46]).

$$\frac{1}{M_n(f)}=\gamma_n(f)=\min_{\substack{q\in P_n\\ q\neq 0}}\frac{\max_{x\in E_n} q(x)\,\sigma_n(x)}{\|q\|}.$$

Thus, to prove our result, it will be sufficient to find polynomials $q_n\in P_n$, where $\|q_n\|$ becomes infinitely large in comparison to $\max_{x\in E_n} q_n(x)\,\sigma_n(x)$, as n increases.

We decompose the set $E_n=\bigcup_{j=1}^m E_n^j$ into sign components

$$E_n^1<E_n^2<\cdots<E_n^m, \qquad \text{i.e.,} \qquad x<y \qquad \text{for all} \quad x\in E_n^j,\ y\in E_n^{j+1},$$

such that $\sigma_n(x)$ is constant on each E_n^j and $m=m(n)$ is minimal. For each n, where $q_n^*\neq q_{n+1}^*$, we have $m(n)=n+2$. Thus, if f is not a polynomial, there exists a subsequence L of $\mathbb{N}_0$ such that $m(n)=n+2$, $n\in L$.

For the sets $E_n=\bigcup_{j=1}^{n+2} E_n^j$, $n\in L$, we define

$$\xi_j=\xi_j(n):=\min E_n^j \qquad \text{and} \qquad \eta_j=\eta_j(n):=\max E_n^j, \qquad 1\leqslant j\leqslant n+2.$$

We follow an argument of H.-P. Blatt [3, p. 46] and consider the following set of problems:

Problem $A(n,k,y)$. Let $k\in\{1,\ldots,n+2\}$ and $y\in E_n^k$ be fixed. Determine $p_n^k\in P_n$ such that

$$-\sigma_n(y)\,p_n^k(y) \qquad \text{is maximal}$$

subject to the condition that

$$\max_{x \in E_n} p_n^k(x)\, \sigma_n(x) \leqslant 1.$$

By [3, Lemma 1], the problem $A(n, k, y)$ has a solution $p_n^k \in P_n$. Moreover, for any solution p_n^k, there exist $n+1$ points $X_n^k = (x_1, ..., x_{k-1}, x_{k+1}, ..., x_{n+2})$ such that

$$x_j \in E_n^j \qquad \text{and} \qquad p_n^k(x_j)\, \sigma_n(x_j) = 1, \qquad \text{for all} \quad 1 \leqslant j \leqslant n+2,\ j \neq k.$$

For the sake of simplicity we avoid noting the index y for p_n^k, X_n^k, as well as n, k, y for the points x_j of X_n^k.

The points in the solution X_n^k of any of the problems $A(n, k, y)$ are ordered in the following way

$$\begin{aligned} &\xi_1 \leqslant x_1 \leqslant \eta_1 < \cdots < \xi_{k-1} \leqslant x_{k-1} \leqslant \eta_{k-1} < \xi_k \leqslant y \\ &y \leqslant \eta_k < \xi_{k+1} \leqslant x_{k+1} \leqslant \eta_{k+1} < \cdots < \xi_{n+2} \leqslant x_{n+2} \leqslant \eta_{n+2}. \end{aligned} \tag{2}$$

Further, since $\sigma_n|_{E_n^j} = -\sigma_n|_{E_n^{j+1}}$, we obtain some relations for the zeros ζ_j of p_n^k.

In case $k=1$ or $k=n+2$, there exist exactly n zeros of p_n^k which are ordered in the following way

$$x_2 < \zeta_2 < x_3 < \cdots < \zeta_{n+1} < x_{n+2}, \qquad k=1, \tag{3}$$

$$x_1 < \zeta_2 < x_2 < \cdots < \zeta_{n+1} < x_{n+1}, \qquad k=n+2. \tag{4}$$

In case $2 \leqslant k \leqslant n+1$, there exist exactly $n-1$ zeros $\zeta_2, ..., \zeta_{k-1}, \zeta_{k+1}, ..., \zeta_{n+1}$ of p_n^k in $[x_1, x_{n+2}]$ which are ordered in the following way

$$x_1 < \zeta_2 < x_2 < \cdots < \zeta_{k-1} < x_{k-1} < y < x_{k+1} < \zeta_{k+1} < \cdots < \zeta_{n+1} < x_{n+2}. \tag{5}$$

Moreover, in this case, there may exist one additional zero $\zeta_0 \notin [x_1, x_{n+2}]$.

2. PROOF OF THE RESULT

We assume that $M_n(f) \leqslant M < \infty$ for all $n \in \mathbb{N}_0$. By Theorem B this implies

$$\|p_n^k\| \leqslant M \tag{6}$$

for all possible $n \in L$, $1 \leqslant k \leqslant n+2$ and $y \in E_n^k$. In particular, the Bernstein inequality [10, p. 118] yields

$$|(p_n^k)'(x)| \leqslant \frac{n}{\sqrt{1-x^2}} M, \qquad x \in [-1, 1]. \tag{7}$$

The proof turns out to be elementary but somewhat technical. Therefore it is split into several lemmas which are implied by our assumption (6) and which will finally lead to a contradiction.

Throughout the proof C and D are used to denote absolute positive constants that depend only on the function f. Whenever involved in estimates for the solutions p_n^k and X_n^k of a problem $A(n, k, y)$ they do in particular not depend on the special problem $A(n, k, y)$ being under consideration. We note that C, D used in different places of the proof may have different values.

In a first step we obtain some control on the distances between the various points induced by the sets E_n^j and the problems $A(n, k, y)$. We will get estimates from above for the distance of any two such points and estimates from below, whenever there exists a point of X_n^k and zero of p_n^k between two such points.

LEMMA 1. *For each $n \in \mathbb{N}$ let*

$$d_n(j) := \frac{\min\{j, n+3-j\}}{n^2}, \qquad 1 \leqslant j \leqslant n+2$$

and

$$d_n(j, v) = d_n(v, j) = \sum_{l=j}^{v} d_n(l), \qquad 1 \leqslant j \leqslant v \leqslant n+2.$$

Then there exist constants $C_1, D_1 > 0$ not depending on $n \in L$, $1 \leqslant k \leqslant n+2$, or on the choice of y in $A(n, k, y)$ such that the following properties hold.

Let $\xi_1, ..., \xi_{n+2}$ and $\eta_1, ..., \eta_{n+2}$ denote the end points of $E_n^1, ..., E_n^{n+2}$.

Further, let p_n^k, $X_n^k = (x_1, ..., x_{k-1}, x_{k+1}, ..., x_{n+2})$ denote the solution of $A(n, k, y)$ and let the zeros of p_n^k be numbered according to (3), (4), (5).

(a) *For all points x_j in X_n^k and any zero $\zeta \in [-1, 1]$ of p_n^k we have*

$$|\eta_j - \xi_j| \leqslant C_1 d_n(j), \qquad 1 \leqslant j \leqslant n+2,$$

$$D_1 d_n(j) \leqslant |x_{j+1} - x_j| \leqslant C_1 d_n(j), \qquad j \in \{1, ..., n+1\} \setminus \{k-1, k\},$$

$$D_1 d_n(j) \leqslant |x_j - \zeta|.$$

(b) *For all points x_j, x_ν in X_n^k and any zero $\zeta_\nu \in [-1, 1]$ of p_n^k we have*

$$D_1 d_n(j, \nu) \leqslant |x_j - x_\nu| \leqslant C_1 d_n(j, \nu),$$
$$j \neq \nu,\ \{j, \nu\} \neq \{k-1, k+1\},$$
$$D_1 d_n(j, \nu) \leqslant |x_j - \zeta_\nu| \leqslant C_1 d_n(j, \nu),$$
$$D_1 d_n(k, \nu) \leqslant |y - \zeta_\nu| \leqslant C_1 d_n(k, \nu),$$
$$D_1 d_n(j, \nu) \leqslant |x_j - \eta_\nu|, |x_j - \xi_\nu|,$$
$$|\nu - j| \geqslant 2,\ \{j, \nu\} \neq \{k-1, k+1\},$$
$$|x_j - \eta_\nu|, |x_j - \xi_\nu| \leqslant C_1 d_n(j, \nu), \qquad 1 \leqslant \nu \leqslant n+2.$$

Proof of Lemma 1. Let μ denote the arcsine distribution of $[-1, 1]$

$$d\mu(x) = \frac{2}{\pi} \frac{1}{\sqrt{1-x^2}}, \qquad x \in [-1, 1].$$

First, we show that for some $C > 0$ we have

$$\mu([-1, \xi_2]), \mu([\eta_{n+1}, 1]), \mu([\eta_{j-1}, \xi_{j+1}]) \leqslant C/n, \qquad 2 \leqslant j \leqslant n+1. \tag{8}$$

If not, we can select such subintervals, say I_n, such that $\limsup_{n \in L} n\mu(I_n) = \infty$. Then, by a slight modification of the proof of [2, Theorem 6], there exist polynomials $q_n \in P_n$ satisfying

$$q_n(x) \neq 0, \qquad x \in I_n,$$

and

$$\liminf_{n \in L} \sup_{x \in [-1, 1] \setminus I_n} \frac{|q_n(x)|}{\|q_n\|} = 0.$$

By Theorem B, this contradicts our assumption (6) on the boundedness of $(M_n(f))_n$.

We have $|p_n^k(x_{j+1}) - p_n^k(x_j)| = 2$ and $|p_n^k(x_j) - p_n^k(\zeta)| = 1$ for all x_j, x_{j+1} in X_n^k and any zero $\zeta \in [-1, 1]$ of p_n^k. Therefore, by (7), we may find some $D > 0$ such that

$$\mu([x_j, x_{j+1}]), \mu([x_j, \zeta]) \geqslant D/n. \tag{9}$$

To derive the estimates stated in Lemma 1, we consider the transformation $x = \cos \varphi$, $\varphi \in [0, \pi]$. The inequalities (8), (9) and the interlacing properties given in (2), (3), (4), (5) imply estimates for the angles belonging to the various points.

We shall only give the idea for $|x_j - x_{j+1}|$, where $x_j, x_{j+1} \in [-1, 0]$ and $j+1 < k$. Let $x_j = \cos(\varphi_j) < x_{j+1} = \cos(\varphi_{j+1})$, $\pi/2 \leqslant \varphi_{j+1} < \varphi_j \leqslant \pi$.

Since

$$\mu([x_1, x_2]) + \cdots + \mu([x_j, x_{j+1}]) \leqslant \mu([-1, x_{j+1}]) \leqslant \mu([-1, \xi_2]) + \cdots + \mu([\xi_j, \eta_{j+1}]),$$

we have $\pi - Dj\pi/n \geqslant \varphi_{j+1} \geqslant \pi - C(j+1)\,\pi/n$.

Since $\mu([x_j, x_{j+1}]) \leqslant \mu([\xi_j, \eta_{j+1}])$, we further have $2C\pi/n \geqslant |\varphi_{j+1} - \varphi_j| \geqslant D\pi/n$.

The estimates for $|x_{j+1} - x_j|$ now follow from

$$|x_j - x_{j+1}| = \left| \int_{\varphi_j}^{\varphi_{j+1}} \sin(t)\, dt \right| \geqslant \frac{2}{\pi} \left| \int_{\varphi_j}^{\varphi_{j+1}} (\pi - t)\, dt \right|$$

and

$$|x_j - x_{j+1}| = \left| \int_{\varphi_j}^{\varphi_{j+1}} \sin(t)\, dt \right| \leqslant \left| \int_{\varphi_j}^{\varphi_{j+1}} (\pi - t)\, dt \right|,$$

with some suitable constants $C_1, D_1 > 0$.

All statements in part (a) can be derived in this manner and the estimates of the second part are a direct consequence of part (a).

LEMMA 2. *There exist constants $C_2, D_2 > 0$ not depending on $n \geqslant 3$ or $1 \leqslant j \leqslant n+2$ such that*

$$\text{(a)} \quad D_2 \log(n) \leqslant \sum_{\substack{v=1 \\ v \neq j}}^{n+2} \frac{d_n(v)}{d_n(j, v)} \leqslant C_2 \log(n),$$

$$\text{(b)} \quad \sum_{\substack{v=1 \\ v \neq j}}^{n+2} \frac{d_n(j)}{d_n(j, v)} \leqslant C_2 \log(n),$$

$$\text{(c)} \quad \sum_{v=2}^{n+1} \frac{d_n(v)}{\sqrt{d_n(1, v)\, d_n(v, n+2)}} \leqslant C_2,$$

$$\text{(d)} \quad \sum_{\substack{v=1 \\ v \neq j}}^{n+2} \frac{d_n(v)^2}{d_n(j, v)^2}, \qquad \sum_{\substack{v=1 \\ v \neq j}}^{n+2} \frac{d_n(v)\, d_n(j)}{d_n(j, v)^2} \leqslant C_2,$$

$$\text{(e)} \quad \left(\prod_{v=3}^{n} \frac{d_n(1, v)\, d_n(n+2, v)}{d_n(v)^2} \right)^{1/2n} \leqslant C_2 n.$$

Proof of Lemma 2. The definition of $d_n(j, v)$ yields that

$$\frac{1}{2n^2}\,|j^2-v^2| \leqslant d_n(j, v) \leqslant \frac{1}{n^2}\,|j^2-v^2|, \qquad 1 \leqslant j \neq v \leqslant \frac{n+3}{2}$$

and

$$\frac{1}{2n^2}\,|(n+3-j)^2-(n+3-v)^2| \leqslant d_n(j, v)$$

$$d_n(j, v) \leqslant \frac{1}{n^2}\,|(n+3-j)^2-(n+3-v)^2|, \qquad \frac{n+3}{2} \leqslant j \neq v \leqslant n+2.$$

(1) We shall prove part (a) only for the case that $1 \leqslant j \leqslant n' := [(n+3)/2]$. It is easy to see that

$$\sum_{\substack{v=1\\ v\neq j}}^{n'} \frac{d_n(v)}{d_n(j, v)} \geqslant \sum_{\substack{v=n'+1\\ v\neq n+3-j}}^{n+2} \frac{d_n(v)}{d_n(j, v)} \geqslant \sum_{v=n'+1}^{n+2} \frac{d_n(v)}{d_n(j, v)} - 1.$$

Therefore, it is sufficient to consider

$$\sum_{\substack{v=1\\ v\neq j}}^{n'} \frac{d_n(v)}{d_n(j, v)} \leqslant 2 \sum_{\substack{v=1\\ v\neq j}}^{n'} \frac{v}{|j^2-v^2|} \leqslant 2 \sum_{\substack{v=1\\ v\neq j}}^{n'} \frac{1}{|j-v|} \leqslant C\log(n') \leqslant C\log(n),$$

for some $C>0$. On the other hand

$$\sum_{\substack{v=1\\ v\neq j}}^{n'} \frac{d_n(v)}{d_n(j, v)} \geqslant \sum_{\substack{v=1\\ v\neq j}}^{n'} \frac{v}{|j^2-v^2|} = \frac{1}{2}\left(\sum_{v=1}^{j-1} \frac{1}{j-v} - \frac{1}{j+v} + \sum_{v=j+1}^{n'} \frac{1}{v-j} + \frac{1}{v+j}\right)$$

$$= \frac{1}{2}\left(\sum_{v=1}^{j-1} \frac{1}{v} - \sum_{v=j+1}^{2j-1} \frac{1}{v} + \sum_{v=1}^{n'-j} \frac{1}{v} + \sum_{v=2j+1}^{n'+j} \frac{1}{v}\right)$$

$$\geqslant \frac{1}{2}\left(\sum_{v=1}^{n'/2} \frac{1}{v} - \sum_{v=j+1}^{2j-1} \frac{1}{v}\right) \geqslant D\log(n') \geqslant D(\log(n)-\log(2)),$$

for some $D>0$, since the negative term remains uniformly bounded for all possible n and j. Part (a) now follows with some suitable $C_2, D_2>0$.

(2) Part (b) may be proved similarly to part (a).

(3) Let $n' := [(n+3)/2)]$. For reasons of symmetry we have

$$\sum_{v=2}^{n+1} \frac{d_n(v)}{\sqrt{d_n(1, v)\, d_n(v, n+2)}} \leqslant 2 \sum_{v=2}^{n'} \frac{d_n(v)}{\sqrt{d_n(1, v)\, d_n(v, n+2)}}.$$

It is easy to see that there exists some $D>0$ such that $d_n(v, n+2) \geqslant D$ for all $1 \leqslant v \leqslant n'$. Thus, we get

$$\leqslant \frac{2}{D^{1/2}} \sum_{v=2}^{n'} \frac{d_n(v)}{\sqrt{d_n(1, v)}} \leqslant \frac{2^{3/2}}{D^{1/2}} \frac{1}{n} \sum_{v=2}^{n'} \frac{v}{\sqrt{v^2-1}} \leqslant C_2,$$

for some suitable $C_2>0$.

(4) We shall prove only the second estimate of part (d) for the case that $1 \leqslant j \leqslant n' := [(n+3)/2]$. Similarly to part (1), it is sufficient to consider the sum

$$\sum_{\substack{v=1 \\ v \neq j}}^{n'} \frac{d_n(v)\, d_n(j)}{d_n(j, v)^2} \leqslant 4 \sum_{\substack{v=1 \\ v \neq j}}^{n'} \frac{vj}{(j^2-v^2)^2} = 4 \sum_{\substack{v=1 \\ v \neq j}}^{n'} \frac{1}{(j-v)^2} \frac{vj}{(j+v)^2}$$

$$\leqslant 4 \sum_{\substack{v=1 \\ v \neq j}}^{n'} \frac{1}{(j-v)^2} \leqslant C,$$

for some suitable $C>0$. This implies the second estimate of part (d) with some suitable $C_2>0$.

(5) Let $n' := [(n+3)/2)]$. For reasons of symmetry we have

$$\prod_{v=3}^{n} \frac{d_n(1, v)\, d_n(n+2, v)}{d_n(v)^2} \leqslant \left(\prod_{v=3}^{n'} \frac{d_n(1, v)\, d_n(n+2, v)}{d_n(v)^2} \right)^2.$$

It is easy to see that there exists some $C>0$ such that $d_n(v, n+2) \leqslant C$ for all $1 \leqslant v \leqslant n+2$ and we get

$$\leqslant C^{2n'} \left(\prod_{v=3}^{n'} \frac{d_n(1, v)}{d_n(v)^2} \right)^2 \leqslant C^{2n'} \left(\prod_{v=3}^{n'} n^2 \frac{v^2-1)}{v^2} \right)^2$$

$$\leqslant C^{2n'} n^{4n'} = C^{2[(n+3)/2]} n^{4[(n+3)/2]},$$

which implies (e) with some suitable $C_2>0$.

Next, we show that the products $\prod_{j \neq k} |y-x_j|$ become relatively small for the solution X_n^k of any of the problems $A(n, k, y)$, $3 \leqslant k \leqslant n$, as $n \in L$ increases.

LEMMA 3. *There exist constants $\delta>0$ and $C_3>0$ not depending on $n \in L$, $3 \leqslant k \leqslant n$, or on the choice of y in $A(n, k, y)$ such that for $X_n^k = (x_1, ..., x_{k-1}, x_{k+1}, ..., x_{n+2})$ we have*

$$\prod_{\substack{j=1 \\ j \neq k}}^{n+2} |y-x_j| \leqslant \frac{C_3}{n^\delta} \frac{|(y-x_{k-1})(y-x_{k+1})|}{d_n(k-1)\, d_n(k+1)} \frac{1}{2^n}.$$

Proof of Lemma 3. For $3 \leqslant k \leqslant n$ we have exactly $n-1$ zeros ζ_j in $[x_1, x_{n+2}]$ ordered in the following way

$$x_1 < \zeta_2 < x_2 < \cdots < \zeta_{k-1} < x_{k-1} < y < x_{k+1} < \zeta_{k+1} < \cdots < \zeta_{n+1} < x_{n+2}.$$

In case p_n^k has exact degree n, there is one additional zero $\zeta_0 \notin [x_1, x_{n+2}]$.

(1) We distinguish between the cases that p_n^k has exact degree n and exact degree $n-1$. If $p_n^k(x) = a_n^k x^{n-1} + \cdots$, $a_n^k \neq 0$, then for the polynomial

$$q(x) := \frac{p_n^k(x)}{a_n^k} \in P_{n-1}$$

we may find n points x_l of X_n^k where q has alternating signs and

$$|q(x_l)| \geqslant \frac{1}{|a_n^k|}.$$

Thus, we must have

$$\frac{1}{|a_n^k|} \leqslant \frac{1}{2^{n-2}}$$

and

$$\begin{aligned} |(y-x_1)(y-x_{n+2})| \prod_{\substack{j=2\\ j\neq k}}^{n+1} |y-\zeta_j| &= |(y-x_1)(y-x_{n+2})| \frac{|p_n^k(y)|}{|a_n^k|} \\ &\leqslant |(y-x_1)(y-x_{n+2})| \frac{M}{|a_n^k|} \\ &\leqslant |(y-x_1)(y-x_{n+2})| \frac{4M}{2^n} \leqslant C \frac{1}{2^n} \end{aligned}$$

for some $C > 0$.

If $p_n^k(x) = a_n^k x^n + \cdots$, $a_n^k \neq 0$, then for the polynomial

$$q(x) := \prod_{\substack{j=2\\ j\neq k}}^{n+1} (x-\zeta_j) = \frac{p_n^k(x)}{a_n^k(x-\zeta_0)} \in P_{n-1}$$

we may find n points x_l of X_n^k where q has alternating signs and

$$|q(x_l)| \geqslant \frac{1}{|a_n^k(x_l-\zeta_0)|} \geqslant \frac{1}{|a_n^k|(|\zeta_0|+1)}.$$

Thus, we must have

$$\frac{1}{|a_n^k|(|\zeta_0|+1)} \leqslant \frac{1}{2^{n-2}}$$

and

$$\begin{aligned}
&|(y-x_1)(y-x_{n+2})| \prod_{\substack{j=2\\ j\neq k}}^{n+1} |y-\zeta_j| \\
&\quad = |(y-x_1)(y-x_{n+2})| \frac{|p_n^k(y)|}{|a_n^k|\,|y-\zeta_0|} \\
&\quad \leqslant |(y-x_1)(y-x_{n+2})| \frac{M}{|a_n^k|\,(|y-\zeta_0|)} \\
&\quad \leqslant |(y-x_1)(y-x_{n+2})| \frac{4M(|\zeta_0|+1)}{|y-\zeta_0|}\,\frac{1}{2^n} \\
&\quad \leqslant C\frac{1}{2^n},
\end{aligned}$$

for some $C>0$ which, in particular, does not depend on the position of $\zeta_0 \notin [x_1, x_{n+2}]$.

(2) We estimate

$$\begin{aligned}
&\left\{\prod_{\substack{j=2\\ j\neq k}}^{n+1} |y-x_j|\right\}\left\{\prod_{\substack{j=2\\ j\neq k}}^{n+1} |y-\zeta_j|\right\}^{-1} \\
&\quad = \frac{|(y-x_{k-1})(y-x_{k+1})|}{|(y-\zeta_{k-1})(y-\zeta_{k+1})|} \prod_{j=2}^{k-2} \frac{|y-x_j|}{|y-\zeta_j|} \prod_{j=k+2}^{n+1} \frac{|y-x_j|}{|y-\zeta_j|} \\
&\quad = \frac{|(y-x_{k-1})(y-x_{k+1})|}{|(y-\zeta_{k-1})(y-\zeta_{k+1})|} \prod_{j=2}^{k-2} \left(1-\frac{|x_j-\zeta_j|}{|y-\zeta_j|}\right) \prod_{j=k+2}^{n+1} \left(1-\frac{|x_j-\zeta_j|}{|y-\zeta_j|}\right).
\end{aligned}$$

By Lemma 1 we have

$$\frac{|(y-x_{k-1})(y-x_{k+1})|}{|(y-\zeta_{k-1})(y-\zeta_{k+1})|} \leqslant \frac{|(y-x_{k-1})(y-x_{k+1})|}{D_1^2 d_n(k-1)\, d_n(k+1)}.$$

Further, Lemma 1 and Lemma 2(a) yield

$$\begin{aligned}
&\prod_{j=2}^{k-2}\left(1-\frac{|x_j-\zeta_j|}{|y-\zeta_j|}\right)\prod_{j=k+2}^{n+1}\left(1-\frac{|x_j-\zeta_j|}{|y-\zeta_j|}\right)\\
&\quad\leqslant \exp\left\{-\sum_{j=2}^{k-2}\frac{|x_j-\zeta_j|}{|y-\zeta_j|}-\sum_{j=k+2}^{n+1}\frac{|x_j-\zeta_j|}{|y-\zeta_j|}\right\}\\
&\quad\leqslant \exp\left\{-\frac{D_1}{C_1}\left(\sum_{j=2}^{k-2}\frac{d_n(j)}{d_n(k,j)}+\sum_{j=k+2}^{n+1}\frac{d_n(j)}{d_n(k,j)}\right)\right\}\\
&\quad\leqslant \exp\left\{-\frac{D_1}{C_1}\left(D_2\log(n)-\frac{d_n(1)}{d_n(k,1)}-\frac{d_n(k-1)}{d_n(k,k-1)}\right.\right.\\
&\qquad\left.\left.-\frac{d_n(k+1)}{d_n(k,k+1)}-\frac{d_n(n+2)}{d_n(k,n+2)}\right)\right\}\\
&\quad\leqslant \frac{C}{n^\delta},
\end{aligned}$$

for some suitable $C, \delta > 0$.

(3) Putting part (1) and part (2) together, we obtain that

$$\prod_{\substack{j=1\\ j\neq k}}^{n+2} |y-x_j| \leqslant \frac{C}{n^\delta}\,\frac{|(y-x_{k-1})(y-x_{k+1})|}{d_n(k-1)\,d_n(k+1)}\,\frac{1}{2^n},$$

for some $C_3 := C > 0$, and Lemma 3 is proved.

In the following Lemmas 4, 5, and 6 we consider the solutions $X_n^1 = (x_2, ..., x_{n+2})$ of the special problems $A(n, 1, y)$ with some arbitrary $y \in E_n^1$, e.g., $y = \xi_1$.

For convenience we put $x_1 := y = \xi_1$. If $\zeta_2, ..., \zeta_{n+1}$ denote the zeros of p_n^1, we obtain

$$\xi_1 = x_1 = y \leqslant \eta_1 < \xi_2 \leqslant \cdots \leqslant \eta_{n+1} < \xi_{n+2} \leqslant x_{n+2} \leqslant \eta_{n+2},$$

and

$$x_1 < x_2 < \zeta_2 < x_3 < \cdots < x_{n+1} < \zeta_{n+1} < x_{n+2}.$$

First, we show that most of the products $\prod_{v\neq j} |x_j - x_v|$ do not become too small for our solution X_n^1, as $n \in L$ increases.

LEMMA 4. *Suppose that* $\varepsilon > 0$. *For* $X_n^1 = (x_2, ..., x_{n+2})$, $x_1 := \xi_1$ *let* $a(n)$ *denote the number of indices* $j \in \{2, ..., n+2\}$ *such that*

$$\prod_{\substack{\nu=1 \\ \nu \neq j}}^{n+2} |x_j - x_\nu| \leqslant \frac{n^{1-\varepsilon}}{2^n}.$$

Then we have

$$\lim_{n \in L} \frac{a(n)}{n} = 0.$$

Proof of Lemma 4. Suppose there exists some $a > 0$ and a subsequence L' of L such that

$$\frac{a(n)}{n} \geqslant a \qquad \text{for all} \quad n \in L'.$$

Then, for $n \in L'$, there exist at least $an/2$ indices $j \geqslant an/2$ such that

$$\prod_{\substack{\nu=1 \\ \nu \neq j}}^{n+2} |x_j - x_\nu| \leqslant \frac{n^{1-\varepsilon}}{2^n},$$

and thus, by Lemma 1,

$$\prod_{\nu \neq j, 1} |x_j - x_\nu| \leqslant \frac{n^{1-\varepsilon}}{2^n} \frac{1}{|x_j - x_1|} \leqslant \frac{n^{1-\varepsilon}}{2^n} \frac{1}{D_1 d_n(1, j)}$$
$$\leqslant \frac{n^{1-\varepsilon}}{2^n} \frac{1}{D_1 d_n(1, [an/2])} \leqslant C \frac{n^{1-\varepsilon}}{2^n},$$

for some $C = C(a) > 0$.

The polynomial $p_n^1(x) = a_n^1 x^n + \cdots$ has exact degree n. Because of the alternation property of p_n^1 at the $n+1$ points $x_2, ..., x_{n+2}$, the Lagrange interpolation formula yields

$$|a_n^1| = \sum_{j=2}^{n+2} \frac{1}{\prod_{\nu \neq j, 1} |x_j - x_\nu|} \geqslant \frac{an}{2} \frac{2^n}{Cn^{1-\varepsilon}} \geqslant \frac{a}{C} n^\varepsilon 2^n.$$

Therefore, we get that $\|p_n^1\| \geqslant |a_n^1| \, 1/2^{n-1}$ becomes unbounded, as $n \in L'$ increases, which contradicts our principal assumption (6). Hence, Lemma 4 is proved.

In the next step we compare the product $\prod_{|\nu - j| \geqslant 2} |x_j - x_\nu|$ to the product of the distances of x_j to $\eta_1, ..., \eta_{j-2}, \xi_{j+2}, ..., \xi_{n+2}$, i.e., to the end-points of E_n^ν, $|\nu - j| \geqslant 2$, which are close to x_j. Obviously, the first product can not be smaller than the second. We show that in average it is larger at most by a factor n, as $n \in L$ increases.

LEMMA 5. *Let* $X_n^1 = (x_2, ..., x_{n+2})$, $x_1 := \xi_1$. *Then there exists a constant* $C_5 > 0$ *not depending on* $n \in L$ *such that*

$$\prod_{j=3}^{n} \left(\left\{ \prod_{\substack{v=1 \\ |v-j| \geq 2}}^{n+2} |x_j - x_v| \right\} \left\{ \prod_{v=1}^{j-2} |x_j - \eta_v| \prod_{v=j+2}^{n+2} |x_j - \xi_v| \right\}^{-1} \right)^{1/n} \leq C_5 n.$$

Proof of Lemma 5. (a) We write every x_v, $1 \leq v \leq n+2$, as a convex combination of ξ_v, η_v

$$x_v = \alpha_v \xi_v + (1-\alpha_v)\,\eta_v, \qquad \alpha_v \in [0, 1].$$

Then we have for $3 \leq j \leq n$

$$\begin{aligned}
&\left\{ \prod_{\substack{v=1 \\ |v-j| \geq 2}}^{n+2} |x_j - x_v| \right\} \left\{ \prod_{v=1}^{j-2} |x_j - \eta_v| \prod_{v=j+2}^{n+2} |x_j - \xi_v| \right\}^{-1} \\
&= \prod_{v=1}^{j-2} \left(1 + \frac{|x_v - \eta_v|}{|x_j - \eta_v|}\right) \prod_{v=j+2}^{n+2} \left(1 + \frac{|x_v - \xi_v|}{|x_j - \xi_v|}\right) \\
&= \prod_{v=1}^{j-2} \left(1 + \frac{\alpha_v(\eta_v - \xi_v)}{|x_j - \eta_v|}\right) \prod_{v=j+2}^{n+2} \left(1 + \frac{(1-\alpha_v)(\eta_v - \xi_v)}{|x_j - \xi_v|}\right) \\
&\leq \exp\left\{ \sum_{v=1}^{j-2} \frac{\alpha_v(\eta_v - \xi_v)}{|x_j - \eta_v|} + \sum_{v=j+2}^{n+2} \frac{(1-\alpha_v)(\eta_v - \xi_v)}{|x_j - \xi_v|} \right\}. \qquad (10)
\end{aligned}$$

On the other hand, since $|x_v - \eta_{v-1}| \geq |x_v - \xi_v| = (1-\alpha_v)(\eta_v - \xi_v)$ and $|x_v - \xi_{v+1}| \geq |x_v - \eta_v| = \alpha_v(\eta_v - \xi_v)$, we have for every $3 \leq j \leq n$,

$$\begin{aligned}
&\left\{ \prod_{\substack{v=1 \\ |v-j| \geq 2}}^{n+2} |x_j - x_v| \right\} \left\{ \prod_{v=1}^{j-2} |x_j - \eta_v| \prod_{v=j+2}^{n+2} |x_j - \xi_v| \right\}^{-1} \\
&\qquad \times \frac{|(x_j - \eta_{j-2})(x_j - \xi_{j+2})|}{|(x_j - x_1)(x_j - x_{n+2})|} \\
&= \prod_{v=2}^{j-2} \frac{|x_j - x_v|}{|x_j - \eta_{v-1}|} \prod_{v=j+2}^{n+2} \frac{|x_j - x_v|}{|x_j - \xi_{v+1}|} \\
&= \prod_{v=2}^{j-2} \left(1 - \frac{|x_v - \eta_{v-1}|}{|x_j - \eta_{v-1}|}\right) \prod_{v=j+2}^{n+1} \left(1 - \frac{|x_v - \xi_{v+1}|}{|x_j - \xi_{v+1}|}\right) \\
&\leq \prod_{v=2}^{j-2} \left(1 - \frac{(1-\alpha_v)(\eta_v - \xi_v)}{|x_j - \eta_{v-1}|}\right) \prod_{v=j+2}^{n+1} \left(1 - \frac{\alpha_v(\eta_v - \xi_v)}{|x_j - \xi_{v+1}|}\right) \\
&\leq \exp\left\{ -\sum_{v=2}^{j-2} \frac{(1-\alpha_v)(\eta_v - \xi_v)}{|x_j - \eta_{v-1}|} - \sum_{v=j+2}^{n+1} \frac{\alpha_v(\eta_v - \xi_v)}{|x_j - \xi_{v+1}|} \right\}. \qquad (11)
\end{aligned}$$

(b) Now, let $\zeta_2, ..., \zeta_{n+1}$ denote the zeros of p_n^1:

$$x_1 < x_2 < \zeta_2 < x_3 < \cdots < x_{n+1} < \zeta_{n+1} < x_{n+2}.$$

The crucial step of the proof will be to replace the sums occuring in the exponential terms above by sums involving the zeros ζ_j.

There exists some $C>0$ such that the following estimates hold

$$\left|\sum_{\nu=1}^{j-2} \frac{\alpha_\nu(\eta_\nu-\xi_\nu)}{|x_j-\eta_\nu|} - \sum_{\nu=2}^{j-1} \frac{\alpha_\nu(\eta_\nu-\xi_\nu)}{|\zeta_j-x_\nu|}\right| \leqslant C,$$

$$\left|\sum_{\nu=2}^{j-2} \frac{(1-\alpha_\nu)(\eta_\nu-\xi_\nu)}{|x_j-\eta_{\nu-1}|} - \sum_{\nu=2}^{j-1} \frac{(1-\alpha_\nu)(\eta_\nu-\xi_\nu)}{|\zeta_j-x_\nu|}\right| \leqslant C,$$

$$\left|\sum_{\nu=j+2}^{n+2} \frac{(1-\alpha_\nu)(\eta_\nu-\xi_\nu)}{|x_j-\xi_\nu|} - \sum_{\nu=j+1}^{n+1} \frac{(1-\alpha_\nu)(\eta_\nu-\xi_\nu)}{|\zeta_j-x_\nu|}\right| \leqslant C,$$

$$\left|\sum_{\nu=j+1}^{n+1} \frac{\alpha_\nu(\eta_\nu-\xi_\nu)}{|x_j-\xi_{\nu+1}|} - \sum_{\nu=j+1}^{n+1} \frac{\alpha_\nu(\eta_\nu-\xi_\nu)}{|\zeta_j-x_\nu|}\right| \leqslant C.$$

We give the computation only for the first difference. By Lemma 1 we have

$$\begin{aligned}
&\left|\sum_{\nu=1}^{j-2} \frac{\alpha_\nu(\eta_\nu-\xi_\nu)}{|x_j-\eta_\nu|} - \sum_{\nu=2}^{j-1} \frac{\alpha_\nu(\eta_\nu-\xi_\nu)}{|\zeta_j-x_\nu|}\right| \\
&\quad \leqslant \frac{|\alpha_1|\,(\eta_1-\xi_1)}{|x_j-\eta_1|} + \frac{|\alpha_{j-1}|\,(\eta_{j-1}-\xi_{j-1})}{|x_j-\zeta_{j-1}|} \\
&\qquad + \sum_{\nu=2}^{j-2} \frac{|\alpha_\nu|\,(\eta_\nu-\xi_\nu)(|x_\nu-\eta_\nu|+|x_j-\zeta_j|)}{|\,(x_j-\eta_\nu)(\zeta_j-x_\nu)|} \\
&\quad \leqslant \frac{C_1 d_n(1)}{D_1 d_n(j,1)} + \frac{C_1 d_n(j-1)}{D_1 d_n(j)} + \frac{C_1^2}{D_1^2} \sum_{\nu=2}^{j-2} \frac{d_n(\nu)(d_n(\nu)+d_n(j))}{d_n(j,\nu)\, d_n(j,\nu)}.
\end{aligned}$$

Lemma 2(d) yields that the difference may be estimated by some $C>0$.

(c) Multiplying the two estimates (10), (11) in (a) and replacing the sums in the exponential terms according to (b) we get

$$\prod_{j=3}^{n} \left(\left\{ \prod_{\substack{\nu=1 \\ |\nu-j|\geqslant 2}}^{n+2} |x_j-x_\nu| \right\} \left\{ \prod_{\nu=1}^{j-2} |x_j-\eta_\nu| \prod_{\nu=j+2}^{n+2} |x_j-\xi_\nu| \right\}^{-1} \right)^2$$

$$\times \left\{ \prod_{j=3}^{n} \frac{|\,(x_j-x_1)(x_j-x_{n+2})|}{|\,(x_j-\eta_{j-2})(x_j-\xi_{j+2})|} \right\}^{-1}$$

$$\leqslant \exp\left\{4C(n-2)+\sum_{j=3}^{n}\left(\sum_{\nu=2}^{j-1}\frac{(2\alpha_\nu-1)(\eta_\nu-\xi_\nu)}{|\zeta_j-x_\nu|}\right.\right.$$
$$\left.\left.+\sum_{\nu=j+1}^{n+1}\frac{(1-2\alpha_\nu)(\eta_\nu-\xi_\nu)}{|\zeta_j-x_\nu|}\right)\right\}.$$

We put $\beta_\nu := (2\alpha_\nu-1)(\eta_\nu-\xi_\nu)$, and thus $|\beta_\nu| \leqslant C_1 d_n(\nu)$, by Lemma 1.

To estimate the sum occurring in the exponential term above we write

$$\left|\sum_{j=3}^{n}\left(\sum_{\nu=2}^{j-1}\frac{\beta_\nu}{|\zeta_j-x_\nu|}+\sum_{\nu=j+1}^{n+1}\frac{-\beta_\nu}{|\zeta_j-x_\nu|}\right)\right|$$
$$=\left|\sum_{\nu=2}^{n-1}\beta_\nu\sum_{j=\nu+1}^{n}\frac{1}{|x_\nu-\zeta_j|}-\sum_{\nu=4}^{n+1}\beta_\nu\sum_{j=3}^{\nu-1}\frac{1}{|x_\nu-\zeta_j|}\right|.$$

Since $x_\nu-\zeta_j<0$, $\nu<j$, and $x_\nu-\zeta_j>0$, $\nu>j$, it follows that

$$\leqslant\sum_{\nu=4}^{n-1}\left|\beta_\nu\left(\sum_{j=\nu+1}^{n}\frac{1}{x_\nu-\zeta_j}+\sum_{j=3}^{\nu-1}\frac{1}{x_\nu-\zeta_j}\right)\right|$$
$$+\sum_{\nu=2}^{3}|\beta_\nu|\sum_{j=\nu+1}^{n}\frac{1}{|x_\nu-\zeta_j|}+\sum_{\nu=n}^{n+1}|\beta_\nu|\sum_{j=3}^{\nu-1}\frac{1}{|x_\nu-\zeta_j|}$$
$$\leqslant\sum_{\nu=4}^{n-1}\left|\beta_\nu\sum_{j=2}^{n+1}\frac{1}{x_\nu-\zeta_j}\right|+\sum_{\nu=4}^{n-1}|\beta_\nu|\left(\frac{1}{|x_\nu-\zeta_{n+1}|}+\frac{1}{|x_\nu-\zeta_\nu|}+\frac{1}{|x_\nu-\zeta_2|}\right)$$
$$+\sum_{\nu=2}^{3}|\beta_\nu|\sum_{j=\nu+1}^{n}\frac{1}{|x_\nu-\zeta_j|}+\sum_{\nu=n}^{n+1}|\beta_\nu|\sum_{j=3}^{\nu-1}\frac{1}{|x_\nu-\zeta_j|}.$$

By Lemma 1, we get

$$\leqslant C_1\sum_{\nu=4}^{n-1}d_n(\nu)\left|\sum_{j=2}^{n+1}\frac{1}{x_\nu-\zeta_j}\right|$$
$$+\frac{C_1}{D_1}\sum_{\nu=4}^{n-1}d_n(\nu)\left(\frac{1}{d_n(\nu,n+1)}+\frac{1}{d_n(\nu)}+\frac{1}{d_n(\nu,2)}\right)$$
$$+\frac{C_1}{D_1}\sum_{\nu=2}^{3}d_n(\nu)\sum_{j=\nu+1}^{n}\frac{1}{d_n(\nu,j)}+\frac{C_1}{D_1}\sum_{\nu=n}^{n+1}d_n(\nu)\sum_{j=3}^{\nu-1}\frac{1}{d_n(\nu,j)}.$$

Lemma 2(a) and (b) yield

$$\leqslant C_1 \sum_{\nu=4}^{n-1} d_n(\nu) \left| \sum_{j=2}^{n+1} \frac{1}{x_\nu - \zeta_j} \right|$$
$$+ \frac{C_1}{D_1} (C_2 \log(n) + (n-4) + C_2 \log(n) + 2C_2 \log(n) + 2C_2 \log(n))$$
$$\leqslant C_1 \sum_{\nu=4}^{n-1} d_n(\nu) \left| \sum_{j=2}^{n+1} \frac{1}{x_\nu - \zeta_j} \right| + Cn,$$

for some $C > 0$.

With the help of (7) and since $|p_n^1(x_\nu)| = 1$, we can now estimate the remaining sum

$$\sum_{\nu=4}^{n-1} d_n(\nu) \left| \sum_{j=2}^{n+1} \frac{1}{x_\nu - \zeta_j} \right| = \sum_{\nu=4}^{n-1} d_n(\nu) \left| \frac{(p_n^1)'(x_\nu)}{p_n^1(x_\nu)} \right|$$
$$\leqslant \sum_{\nu=4}^{n-1} d_n(\nu) \frac{Mn}{\sqrt{1-x_\nu^2}}$$
$$\leqslant \frac{Mn}{D_1} \left(\sum_{\nu=4}^{n-1} \frac{d_n(\nu)}{\sqrt{d_n(\nu, n+2)\, d_n(\nu, 1)}} \right) \leqslant Cn,$$

for some $C > 0$, by Lemma 2(c).

(d) The estimates in (c) yield that for some $C > 0$ we have

$$\prod_{j=3}^{n} \left(\left\{ \prod_{\substack{\nu=1 \\ |\nu-j| \geqslant 2}}^{n+2} |x_j - x_\nu| \right\} \left\{ \prod_{\nu=1}^{j-2} |x_j - \eta_\nu| \prod_{\nu=j+2}^{n+2} |x_j - \xi_\nu| \right\}^{-1} \right)^{1/n}$$
$$\leqslant C \left\{ \prod_{j=3}^{n} \frac{|(x_j - x_1)(x_j - x_{n+2})|}{|(x_j - \eta_{j-2})(x_j - \xi_{j+2})|} \right\}^{1/2n}$$
$$\leqslant C \left\{ \prod_{j=3}^{n} \frac{|(x_j - x_1)(x_j - x_{n+2})|}{|(x_j - x_{j-1})(x_j - x_{j+1})|} \right\}^{1/2n}$$
$$\leqslant C \frac{C_1}{D_1} \left\{ \prod_{j=3}^{n} \frac{d_n(j, 1)\, d_n(j, n+2)}{d_n(j)\, d_n(j)} \right\}^{1/2n}$$
$$\leqslant C \frac{C_1}{D_1} C_2 n,$$

by Lemma 1 and Lemma 2(e). Putting $C_5 := CC_2 C_1 / D_1$, Lemma 5 is proved.

With the help of Lemma 5 we get

LEMMA 6. *Suppose that* $\varepsilon > 0$. *For* $X_n^1 = (x_2, ..., x_{n+2})$, $x_1 := \xi_1$ *let* $b(n)$ *denote the number of indices* $3 \leqslant j \leqslant n$ *such that*

$$\prod_{\nu=1}^{j-2} |x_j - \eta_\nu| \prod_{\nu=j+2}^{n+2} |x_j - \xi_\nu| \geqslant \frac{1}{n^{1+\varepsilon}} \prod_{\substack{\nu=1 \\ |\nu-j| \geqslant 2}}^{n+2} |x_j - x_\nu|.$$

Then we have

$$\liminf_{n \in L} \frac{b(n)}{n} > 0.$$

Proof of Lemma 6. By Lemma 5 we have

$$\begin{aligned} C_5 n &\geqslant \prod_{j=3}^{n} \left(\left\{ \prod_{\substack{\nu=1 \\ |\nu-j| \geqslant 2}}^{n+2} |x_j - x_\nu| \right\} \left\{ \prod_{\nu=1}^{j-2} |x_j - \eta_\nu| \prod_{\nu=j+2}^{n+2} |x_j - \xi_\nu| \right\}^{-1} \right)^{1/n} \\ &\geqslant (n^{1+\varepsilon})^{(n-2-b(n))/n}, \end{aligned}$$

which leads to a contradiction if we assume that $\liminf_{n \in L} (b(n)/n) = 0$. Hence, Lemma 6 is proved.

Now we are in a position to complete the proof of the theorem. We take $\delta > 0$ from Lemma 3 and choose $\varepsilon = \delta/4$ in Lemma 4 and Lemma 6. Since $\lim_{n \in L} a(n)/n = 0$ and $\liminf_{n \in L} b(n)/n > 0$ we may find for each $n \in L$, n sufficiently large, some index $k = k(n) \in \{3, ..., n\}$ such that for $X_n^1 = (x_2, ..., x_{n+2})$, $x_1 := \xi_1$,

$$\prod_{\substack{\nu=1 \\ \nu \neq k}}^{n+2} |x_k - x_\nu| \geqslant \frac{n^{1-\varepsilon}}{2^n}$$

and

$$\begin{aligned} &\prod_{\nu=1}^{k-2} |x_k - \eta_\nu| \prod_{\nu=k+2}^{n+2} |x_k - \xi_\nu| \\ &\quad \geqslant \frac{1}{n^{1+\varepsilon}} \prod_{\substack{\nu=1 \\ |\nu-k| \geqslant 2}}^{n+2} |x_k - x_\nu| \\ &\quad = \frac{1}{n^{1+\varepsilon}} \frac{1}{|(x_k - x_{k-1})(x_k - x_{k+1})|} \prod_{\substack{\nu=1 \\ \nu \neq k}}^{n+2} |x_k - x_\nu|. \end{aligned}$$

By Lemma 1 we get

$$\prod_{\nu=1}^{k-2} |x_k-\eta_\nu| \prod_{\nu=k+2}^{n+2} |x_k-\xi_\nu| \geqslant \frac{1}{n^{2\varepsilon}} \frac{D_1^2}{d_n(k-1)\, d_n(k+1)} \frac{1}{2^n}$$
$$= \frac{1}{n^{\delta/2}} \frac{D_1^2}{d_n(k-1)\, d_n(k+1)} \frac{1}{2^n}.$$

For $k=k(n)$ we now consider the solutions $X_n^k=(x_1', ..., x_{k-1}', x_{k+1}', ..., x_{n+2}')$ of the problems $A(n, k, x_k)$, i.e., we choose $y=x_k$, where x_k comes from the solution X_n^1 of $A(n, 1, \xi_1)$. We then obtain

$$\prod_{\substack{j=1\\ j\neq k}}^{n+2} |x_k-x_j'| \geqslant |(x_k-x_{k-1}')(x_k-x_{k+1}')| \prod_{j=1}^{k-2} |x_k-\eta_j| \prod_{j=k+2}^{n+2} |x_k-\xi_j|$$
$$\geqslant \frac{D_1^2}{n^{\delta/2}} \frac{|(x_k-x_{k-1}')(x_k-x_{k+1}')|}{d_n(k-1)\, d_n(k+1)} \frac{1}{2^n}.$$

But this contradicts Lemma 3 for large $n\in L$. Therefore, our assumption

$$M_n(f)\leqslant M<\infty, \qquad \text{for all} \quad n\in L$$

cannot hold and the theorem is proved.

REFERENCES

1. M. W. Bartelt and H. W. McLaughlin, Characterizations of strong unicity in approximation theory, *J. Approx. Theory* **9** (1973), 255–266.
2. M. W. Bartelt and D. Schmidt, On Poreda's problem for strong unicity constants, *J. Approx. Theory* **33** (1981), 69–79.
3. H.-P. Blatt, Exchange algorithms, error estimations and strong unicity in convex programming and Chebychev approximation, *in* "NATO ASI Ser. C Math. Phys. Sci.," Vol. 136, Kluwer Academic, Dordrecht, 1984.
4. R. Grothmann, "Zur Größenordnung der starken Eindeutigkeitskonstanten von holomorphen Funktionen," Dissertation an der Katholischen Universität Eichstätt, 1986.
5. R. Grothmann, On the real CF-method for polynomial approximation and strong unicity constants, *J. Approx. Theory* **55** (1988), 86–103.
6. M. S. Henry and J. A. Roulier, Lipschitz and strong unicity constants for changing dimension, *J. Approx. Theory* **32** (1978), 85–94.
7. M. S. Henry and J. J. Swetits, Precise orders of strong unicity constants for a class of rational functions, *J. Approx. Theory* **32** (1981), 292–305.
8. M. S. Henry, J. J. Swetits, and S. Weinstein, On extremal sets and strong unicity constants for certain C^∞-functions, *J. Approx. Theory* **37** (1983), 155–174.
9. A. Kroó, The Lipschitz constant of the operator of best uniform approximation, *Acta Math. Acad. Sci. Hungar.* **35** (1980), 279–292.

10. J. P. Natanson, "Konstruktive Funktionentheorie," Akademie-Verlag, Berlin, 1955.
11. D. J. Newman and H. S. Shapiro, Some theorems on Chebyshev approximation, *Duke Math. J.* **30** (1963), 673–682.
12. S. J. Poreda, Counterexamples in best approximation, *Proc. Amer. Math. Soc.* **56** (1976), 167–171.
13. D. Schmidt, On an unboundedness conjecture for strong unicity constants, *J. Approx. Theory* **24** (1978), 216–223.